Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of Sound and Vibration 266 (2003) 261–279

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Normal form computation without central manifold reduction

A.Y.T. Leung[a,*], Q.C. Zhang[b]

[a] Department of Building and Construction, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong
[b] School of Mechanical Engineering, Tiianjin University, People's Republic of China

Received 5 July 2001; accepted 11 September 2002

Abstract

An efficient method for calculating the normal form and the associated non-linear transformations for the semi-simple case without central manifold reduction is given in this paper. The one-step transformation concept is adopted for easy programming. This method can be used to calculate high order normal forms of high-dimensional ordinary differential equations of non-linear oscillators. A program in MATHEMATICA language is designed to perform the calculation. Three examples are given in order to verify the method and to show the efficiency of the program.

1. Introduction

The normal form method has been widely used in the fields of dynamical systems, ordinary differential equations (ODE) and non-linear vibration. For examples, Kern and Maitz [1] applied to flutter prediction. Khajepour et al. [2] designed modal coupling controllers using the normal form. Fredriksson and Nordmark [3] found the normal form of impact oscillators. Yu and Bi [4] analyzed a double pendulum. Yu and Zhang [5] studied a thin plate.

There were four primary methods of calculating normal form. Wang [6] made a summary from the mathematician's view point and gave a detailed introduction to the three basic methods of calculating the normal form of ODE. The three methods are the matrix representation method, the adjoint operator method and the method based on the representation theory of sl(2, **R**). The fourth method of perturbation was used by Nayfeh [7], Yu [8] and Leung and Zhang [9,10]. Recently, a number of other important new and well-established methods [11,12] namely. Liapunov–Schmidt reduction, succession functions and intrinsic harmonic balance have been published to determine normal forms.

*Corresponding author.
E-mail address: andrew.leung@cityu.edu.hk (A.Y.T. Leung).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(02)01626-7

Normal form theory plays an important role in the study of the dynamical behavior of non-linear systems near the dynamic equilibrium points because it greatly simplifies the analysis and formulations. This simple form can be used conveniently for analyzing the dynamical behavior of the original system in the vicinity of the critical equilibrium. However, it is not a simple task to calculate the normal form for a given set of ODE. In particular, it is difficult to derive explicit formulas of a normal form in terms of the coefficients of the original ODEs. A crucial part in computing normal forms is to find the coefficients of the normal forms and the associated non-linear transformations efficiently. The algebraic manipulations become very involved as the order of normal forms and the dimension of ODE increase. Thus, symbolic computations using languages such as MAPLE, MATHEMATICA and MACSYMA have been introduced to compute the normal forms. For examples, Chen and Zhang [13] presented a symbolic computer program using REDUCE for computing the explicit coefficients of the normal form at Hopf bifurcation. Leung and Zhang [10] gave a combined method of normal form and averaging which takes the advantages of each theory to find the higher order averaging equations for normal forms via MATHEMATICA. Yu [8] gave a perturbation method of multiple scales to calculate normal form via MAPLE. Bi and Yu [14] introduced a method to calculate normal form for semi-simple cases via MAPLE. Zhang et al. [11,12] presented a new procedure for obtaining high order normal forms and the associated coefficients via MAPLE.

Normal form is usually applied in conjunction with the central manifold theory [14,15–18]. Earlier work which produces normal forms without first achieving reductions through the application of central manifold theory can be found in Refs. [19,20] dealing with two-dimensional systems. More recently, high-dimensional systems have also been considered in Ref. [21].

A method of calculating normal form without central manifold reduction is given in this paper. Fewer transformations are used in this method. The present method is different from the averaging technique in that normal forms are obtained by applying a sequence of near-identity transformations. It is also different from the other normal form methods by giving non-linear algebraic equations, but not the differential equations, between the variables associated with center manifold and stable manifold. Although the stable manifold does not change the dimension of the central manifold, it does affect the coefficients of the normal form of the same dimension.

Three examples are given to illustrate some of the advantages and to verify the correctness of the proposed approach.

2. Normal form without central manifold reduction

Consider the system of ODE

$$\dot{x} = Ax + f(x, \varepsilon), \quad x \in R^n, \quad f(x, \varepsilon) : R^n \to R^n, \tag{1}$$

where an over-dot denotes differentiation with respect to time. A dimensionless symbol ε is introduced here to denote the degree k of the homogeneous polynomial so that if $p_k(y) \in H_n^k, k = 2, \ldots, n$ is an n-variable homogeneous polynomial of degree k, then $p_k(\varepsilon y) = \varepsilon^k p_k(y)$.

Suppose that the system contains stable and central manifolds only, i.e., there is no eigenvalue with positive real part. Let the eigenvalues of A with zero real parts be $J_0 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_0})$, and those with negative real parts be $J_1 = \mathrm{diag}(\lambda_{n_0+1}, \lambda_{n_0+2}, \ldots, \lambda_n)$. One can write $J = \mathbf{T}^{-1}A\mathbf{T} = \mathrm{diag}(J_0, J_1)$, where $\mathbf{T}$ is the eigenmatrix of A. Further, let $y = (w, v)^{\mathrm{T}} = T^{-1}x$, where w and v are the variables associated with the eigenvalues of J with zero real parts and with negative real parts, respectively. Then, Eq. (1) can be rewritten as

$$\begin{bmatrix} \dot{w} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} + \begin{bmatrix} f_1(w, v) \\ f_2(w, v) \end{bmatrix} \tag{2}$$

and one has the following theorem.

Theorem 1. *If the linear part of* Eq. (1) *is in diagonal form (semi-simple case), then the normal forms are combinations of the resonant monomials* [18].

Therefore, the normal forms of Eq. (1) are combinations of the resonant monomials, which contain only those terms that satisfy

$$\langle m, \lambda \rangle - \lambda_S = \sum_{i=1}^{n} m_i \lambda_i - \lambda_S = 0, \tag{3}$$

where m_i is a positive integer, $s = 1, 2, \ldots, n$.

The method of calculating normal form without central manifold reduction can be used to calculate the normal form of Eq. (1) for the following reasons:

(a) If Eq. (3) is satisfied for s less than or equal to n_0, then Eq. (3) does not contain λ_i terms for i great than n_0. Otherwise Eq. (3) cannot be satisfied as m_i is always greater than zero. The real part of λ_i is less than zero and the real part of λ_S is always equal to zero. So the normal forms of the first n_0 equation contain only the central manifold terms.
(b) If Eq. (3) is satisfied for s greater than n_0, then the corresponding normal forms are combinations of the central and stable manifolds.

3. Solution for the transformation

To transform Eq. (2) into its normal form, a series of almost identical non-linear co-ordinate transformations is required. Let $ad_J^k[p_k(y)] = D_y p_k J y - J p_k$, where D_y denotes partial differentiation with respect to y and Im be the image.

The following theorem has been proved in Ref. [6].

Theorem 2. *Let C_k be the complementary subspaces to* $\mathrm{Im}\, ad_J^k$ *in H_n^k for $k = 2, \ldots, n$. There exist a series of near identity non-linear transformations*

$$x = Jy + p_k(y), \quad p_k(y) \in H_n^k, \quad k = 2, \ldots, n, \tag{4}$$

such that system (2) *reduces to its normal form*

$$\dot{y} = Jy + C(y), \tag{5}$$

where $C(y)$ contains only the resonant terms.

Alternatively, Nayfeh [7] suggested a one-step method transforming Eq. (2) into its normal form based on perturbation. The transformation is written as

$$x = y + P(y), \tag{6}$$

and he showed that the normal form obtained is in agreement with the classical method. In this paper, the one-step transformation method is used. Substituting Eq. (6) into Eq. (2), one obtains

$$(I + \mathbf{D}_y\mathbf{P})\dot{y} = J(y + P) + f(y + P, \varepsilon), \tag{7}$$

where $\mathbf{D}_y\mathbf{P}$ is the Jacobian matrix of $P(y)$. A conventional approach for solving Eq. (7) is to use Taylor's series

$$[I + \mathbf{D}_y\mathbf{P})]^{-1} = I - \mathbf{D}_y\mathbf{P} + (\mathbf{D}_y\mathbf{P})^2 - (\mathbf{D}_y\mathbf{P})^3 + \cdots$$

and substitute it back into Eq. (7) to obtain

$$\begin{aligned}\dot{y} &= [I + \mathbf{D}_y\mathbf{P}]^{-1}[J(y + P) + f(y + P, \varepsilon)] \\ &= [I - \mathbf{D}_y\mathbf{P} + (\mathbf{D}_y\mathbf{P})^2 - (\mathbf{D}_y\mathbf{P})^3 + \cdots][Jy + JP + f(y + P, \varepsilon)].\end{aligned} \tag{8}$$

The difficulties involved in the above procedure are not only that one has to handle a large amount of algebraic manipulations, but also that it is hard to know the number of terms chosen in Eq. (8) for a given order of normal forms. In order to overcome the difficulties due to the computation of the inverse of $(I + \mathbf{D}_y\mathbf{P})$, an alternative approach [22] may be used. It is achieved as follows. First substituting Eq. (5) into Eq. (7), one has

$$[I + \mathbf{D}_y\mathbf{P}][Jy + C(y)] = J(y + P) + f(y + P, \varepsilon). \tag{9}$$

Rearranging, one obtains

$$\mathbf{D}_y\mathbf{P}Jy - JP = F(y, \varepsilon) - D_yPC(y) - C(y), \tag{10}$$

where $F(y, \varepsilon) = f(y + P, \varepsilon)$.
Eq. (10) is all one needs for computing the normal forms and the associated non-linear transformations by iteration. If one can find $P(y)$ and $C(y)$ from Eq. (10), one then has obtained the normal forms and the associated non-linear transformations.

In general, however, closed-form solutions of Eq. (10) cannot be found. Thus, an approximate solution may be assumed in the form of

$$P(y) = \sum_{m\geqslant 2} p_m(y) = \sum_{m\geqslant 2} \varepsilon^{m-1} \sum_{m_1+m_2+\cdots+m_n=m} p_{m_1m_2,\ldots,m_n} y_1^{m_1} y_2^{m_2}, \ldots, y_n^{m_n}, \tag{11}$$

$$C(y) = \sum_{m\geqslant 2} \varepsilon^{m-1} c_m(y) = \sum_{m\geqslant 2} \varepsilon^{m-1} \sum_{m_1+m_2+\cdots+m_n=m} c_{m_1m_2\cdots m_n} y_1^{m_1} y_2^{m_2}, \ldots, y_n^{m_n}, \tag{12}$$

$$F(y, \varepsilon) - \mathbf{D}_y\mathbf{P}C(y) = \sum_{m\geqslant 2} f_m(y) = \sum_{m\geqslant 2}\left(\sum_{m_1+m_2+\cdots+m_n=m} f_{m_1m_2,\ldots,m_n} y_1^{m_1} y_2^{m_2}, \ldots, y_n^{m_n}\right). \tag{13}$$

Then $D_y p_m(y)Jy$ can be written as

$$\begin{aligned}
&\left(D_y \sum_{m_1+m_2+\cdots+m_n=n} p_{m_1m_2,\ldots,m_n} u_1^{m_1} u_2^{m_2} u_3^{m_3}, \ldots, u_n^{m_n}\right)(Jy) \\
&\quad = \left(\sum_{m_1+m_2+\cdots+m_n=n} D_y (p_{m_1m_2,\ldots,m_n} y_1^{m_1} y_2^{m_2} y_3^{m_3}, \ldots, y_n^{m_n})\right)(Jy) \\
&\quad = \sum_{m_1+m_2+\cdots+m_n=n} \left(\sum_{i=1}^{n} \frac{\partial}{\partial y_i} (p_{m_1m_2,\ldots,m_n} y_1^{m_1} y_2^{m_2} y_3^{m_3}, \ldots, y_n^{m_n}) \lambda_i y_i\right) \\
&\quad = \sum_{m_1+m_2+\cdots+m_n=n} (m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n) p_{m_1m_2,\ldots,m_n} y_1^{m_1} y_2^{m_2} y_3^{m_3}, \ldots, y_n^{m_n} \\
&\quad = \sum_{m_1+m_2+\cdots+m_n=n} \lambda_0 p_{m_1m_2,\ldots,m_n} y_1^{m_1} y_2^{m_2} y_3^{m_3}, \ldots, y_n^{m_n}, \qquad (14)
\end{aligned}$$

where

$$\lambda_0 = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n.$$

Substituting Eqs. (11)–(14) into Eq. (10) and equating the coefficients of ε^{m-1} on both sides of Eq. (10), one has

$$(\lambda_0 I - J) p_{m_1m_2,\ldots,m_n} = f_{m_1m_2,\ldots,m_n} - c_{m_1m_2,\ldots,m_n}. \qquad (15)$$

This is a simple algebraic equation for calculating the normal form and the associated non-linear transformation. The solvability of Eq. (15) depends on the singularity of the matrix $\mathbf{A}_0 = (\lambda_0 I - J)$.

Let $\lambda(\mathbf{A}_0)$ denote the eigenvalues of $\mathbf{A}_0$. The non-resonant condition is $\langle m, \lambda \rangle - \lambda_s = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n - \lambda_s \neq 0$. In this case, $\lambda(\mathbf{A}_0) \neq 0$ implies that $\mathbf{A}_0$ is non-singular. Therefore, $p_{m_1m_2,\ldots,m_n}$ can always be determined for a given $c_{m_1m_2,\ldots,m_n}$. In order to obtain a normal form for this particular order as simple as possible, one may choose $c_{m_1m_2,\ldots,m_n} = 0$, and thus,

$$c_{m_1m_2,\ldots,m_n} = 0, \quad p_{m_1m_2,\ldots,m_n} = \mathbf{A}_0^{-1} f_{m_1m_2,\ldots,m_n}. \qquad (16, 17)$$

The resonant condition is $\langle m, \lambda \rangle - \lambda_s = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_n\lambda_n - \lambda_s = 0$. In this case, $\lambda(A_0) = 0$. A_0 is singular and thus, $p_{m_1m_2\cdots m_n}$cannot be uniquely determined from Eq. (15) for a given $c_{m_1m_2,\ldots,m_n}$.

The Fredholm theorem [17] states that:

Theorem 3 (Fredholm). *Let ad_A be a linear operator in a finite dimensional inner product space V.* Suppose *ad_A^* is the adjoint operator of ad_A. Then*

$$V = \operatorname{Im} ad_A \oplus Ker\, ad_A^*,$$

where $\operatorname{Im} ad_A$ *is the image of ad_A and $Ker\, ad_A^*$ is the null space of ad_A^*.*

According to Theorem 2, the procedure of finding the coefficients of $p_{m_1m_2,\dots,m_n}$ and $c_{m_1m_2,\dots,m_n}$ for the resonant case is as follows.

Suppose $c^s_{m_1m_2,\dots,m_n}$ is the resonant term, then it will appear in the normal form according to normal form theory and $c^s_{m_1m_2,\dots,m_n} \in Ker\, \mathbf{A}_0$. Suppose the linear complementary operator of $\mathbf{A}_0$ is $\mathbf{A}^*$, and the number of zero eigenvalues of $\mathbf{A}_0$ is n_s, so that $\mathbf{A}_0 = \mathrm{diag}(0, 0, \dots, 0, \lambda_{n_s+1}, \lambda_{n_s+2}, \dots, \lambda_n)$ and $\mathbf{A}^* = \mathrm{diag}(1, 1, \dots, 1, 0, 0, \dots, 0)$. Where the number of 1's is n_s and the number of 0's is $n - n_s$. Consequently, $\bar{\mathbf{A}} = \mathbf{A}_0 + \mathbf{A}^*$ is non-singular, and Eq. (15) becomes

$$c_{m_1m_2,\dots,m_n} = \mathbf{A}^*\bar{\mathbf{A}}^{-1} f_{m_1m_2,\dots,m_n}, \quad p_{m_1m_2\cdots m_n} = \bar{\mathbf{A}}^{-1}(f_{m_1m_2,\dots,m_n} - c_{m_1m_2,\dots,m_n}). \qquad (18, 19)$$

Eqs. (16)–(19) are all one needs to calculate the normal forms and the corresponding non-linear transformations.

4. Outline of the MATHEMATICA program

A computer program has been designed to compute the normal forms of a set of ODE of any dimension up to nine. The dimension is set for convenience and is limited by the capacity of the computer only. The symbolic computer programs, including the source code, written in MATHEMATICA, are listed on the JSV + website. The program is constructed in the following manner.

(1) *Create the input file*:

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n.$$

One should input $f[i]$ with the state variable $x[i]$ and the degree of polynomials ε and save them in a working subfolder.

(2) *Compute the normal form and its corresponding non-linear transformations*:

1. Set the input and output file path in the first line.
2. Read the input file in the second line.
3. Transform the functions from real co-ordinate form to complex co-ordinate form.
4. Calculate the normal form and its corresponding non-linear transformation in complex form.
5. Transform the normal form into real co-ordinate form.
6. Transform the normal form into polar co-ordinate form.

(3) *The feature of the program*

Four functions are defined, which make the program more efficient and simpler.

(a) Jacobi[funs_List,vars_List]: = Outer[D,funs,vars]. This function is used to calculate the Jacobian matrix of funs about vars.

(b) Co3[i_]: = D[b[k,nn],{u[i],nk[i]}]/nk[i]! This function is used to calculate the coefficient b[k,nn] about the nk[i]-th of u[i].

(c) ne[k_]: = j-Sum[nk[i],{i,1,k-1}]. The combined use of the above two functions and an evaluating command, Table[ne[i] = 0, {I,n,8}], makes the present program much simpler than that of Ref. [14]. It possible to combine several individual program blocks in one. The main program is reduced from three pages to one page despite the fact that the dimension of the function in the present program is higher than that of Ref. [14]. Also, one only needs to add

2($m-9$) sentences in the program if one wants to study an m dimension system, when m is higher than nine. The program can be easily extended to calculate very high-dimension normal form.

(d) conjug = Complex[a_,b_]:- > Complex[a,-b]. This function is used to calculate the conjugate of complex functions in symbolic form.In order to calculate the higher order normal form, one should substitute $y = y + P(y) = y + \Sigma p_i(y)$ into the function of $f(x)$ directly. The term that is higher than the order that one needs to calculate is eliminated by an intelligent judgement in the substituting process. In this way, the program becomes more efficient.The arrangement sequence of the variable of normal forms in polar-co-ordinate form is determined by its corresponding eigenvalues. First the ones with negative real eigenvalues and then the ones with zero eignenvalues, which are followed by the ones with conjugate complex eigenvalues and the ones with pure imaginary eigenvalues.

5. Examples

All six examples in Ref. [14] were calculated in order to verify the presented method and to show the efficiency of the program. The results show that the normal forms on the central manifold are quite the same. Ref. [17] gives only this kind of normal form. Only three of the six examples are given below.

5.1. Example 1

Calculate the fifth order normal form of the following five-dimensional system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \varepsilon \begin{bmatrix} x_1^2 - x_1 x_3 \\ x_2^2 + x_1 x_4 + \varepsilon x_2^3 \\ x_1^2 \\ x_1^2 \\ x_2^2 \end{bmatrix}.$$

The Jacobian matrix of this system evaluated at the equilibrium $x_k = 0,\ k = 1, 2, \dots, 5$ has the eigenvalues $\pm i, -1$ and $-1 \pm i$. Two of them have zero real parts. The center manifold is two dimensional.

The input file is:

```
n = 5;
norder = 5;
f[1] = x[2] + epsilon (x[1]^2–x[1] x[3]);
f[2] = –x[1] + epsilon (x[2]^2 + x[1] x[4] + epsilon x[2]^3);
f[3] = –x[3] + epsilon x[1]^2;
f[4] = –x[4] + x[5] + epsilon x[1]^2;
f[5] = –x[4]–x[5] + epsilon x[2]^2;
```

The normal form of this system up to order five in polar co-ordinates is:

$$\dot{u}_1 = -u_1 + \frac{29}{120}\varepsilon^3\cos(\theta_1-\theta_2)r_1r_2^3 - \frac{1}{10}\varepsilon^3\sin(\theta_1-\theta_2)r_1r_2^3 + \frac{3}{5}r_2^2u_1 + \frac{619}{750}\varepsilon^4r_2^4u_1$$

$$\dot{r}_1 = -r_1 + \frac{1}{20}\varepsilon^2r_1r_2^2 + \frac{143111}{612000}\varepsilon^4r_1r_2^4 + \frac{137}{4800}\varepsilon^4\cos 2(\theta_1-\theta_2)r_1r_2^4$$
$$-\frac{1229}{4800}\varepsilon^4\sin 2(\theta_1-\theta_2)r_1r_2^4$$
$$-\frac{7}{60}\varepsilon^3\cos(\theta_1-\theta_2)r_2^3u_1 - \frac{1}{4}\varepsilon^3\sin(\theta_1-\theta_2)r_2^3u_1$$

$$\dot{\theta}_1 = 1 + \frac{3}{20}\varepsilon^2r_2^2 - \frac{4271}{51000}\varepsilon^4r_2^4 - \frac{1229}{4800}\varepsilon^4\cos 2(\theta_1-\theta_2)r_2^4 - \frac{137}{4800}\varepsilon^4\sin 2(\theta_1-\theta_2)r_2^4$$
$$-\frac{1}{4r_1}\cos(\theta_1-\theta_2)\varepsilon^3r_2^3u_1 + \frac{7}{60r_1}\varepsilon^3\sin(\theta_1-\theta_2)r_2^3u_1$$

$$\dot{r}_2 = \frac{3}{40}\varepsilon^2r_2^3 - \frac{14867}{68000}\varepsilon^4r_2^5, \quad \dot{\theta}_2 = 1 - \frac{7}{12}\varepsilon^2r_2^2 + \frac{5691403}{14688000}\varepsilon^4r_2^4.$$

5.2. Example 2

Calculate the fifth order normal form of the following equation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix} = \begin{bmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&-1&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&1\\0&0&0&0&-1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\\dot{x}_6\end{bmatrix} + \varepsilon\begin{bmatrix}-(x_1-x_2-x_4)^2\\-(x_1-x_2+x_5)^2\\-(x_1-x_2+x_4)^2\\(x_1-x_5)^2\\(x_1-x_4)^2\\(x_2+x_5)^2\end{bmatrix},$$

whose Jacobian matrix evaluated at the origin involves the eigenvalues 0, $\pm i$, -1 and $-1\pm i$.
The normal form has a three-dimensional central manifold. The input file is:

```
n = 6;
norder = 5;
f[1] = –epsilon (x[1]–x[2]–x[4])ˆ2;
f[2] = x[3]–epsilon (x[1]–x[2] + x[5])ˆ2;
f[3] = –x[2]–epsilon (x[2]–x[3] + x[4])ˆ2;
f[4] = –x[4] + epsilon (x[1]–x[5])ˆ2;
f[5] = –x[5] + x[6] + epsilon (x[1]–x[4])ˆ2;
f[6] = –x[5]–x[6] + epsilon (x[2] + x[5])ˆ2;
```

The normal form in polar co-ordinates is:

$$\begin{aligned}\dot{u}_1 = &-u_1 + \frac{11}{12}\varepsilon^3\cos(\theta_1-\theta_2)r_1r_2^3 + \frac{163}{60}\varepsilon^3\sin(\theta_1-\theta_2)r_1r_2^3 - \frac{79}{40}\varepsilon^4r_2^4u_1\\ &-2\varepsilon^2\cos(\theta_1-\theta_2)r_1r_2u_4 - \frac{7789}{900}\varepsilon^4\cos(\theta_1-\theta_2)r_1r_2^3u_4 + 2\varepsilon^2\sin(\theta_1-\theta_2)r_1r_2u_4\\ &+\frac{4601}{450}\varepsilon^4\sin(\theta_1-\theta_2)r_1r_2^3u_4 - 7\varepsilon^3r_2^2u_1u_4 - \frac{3}{5}\varepsilon^3\cos(\theta_1-\theta_2)r_1r_2u_4^2\\ &-\frac{39}{5}\varepsilon^3\sin(\theta_1-\theta_2)r_1r_2u_4^2 - 4\varepsilon^2u_1u_4^2 - \frac{303}{5}\varepsilon^4r_2^2u_1u_4^2 + \frac{1163}{50}\varepsilon^4\cos(\theta_1-\theta_2)r_1r_2u_4^3\\ &-\frac{991}{50}\varepsilon^4\sin(\theta_1-\theta_2)r_1r_2u_4^3 - 2\varepsilon^3u_1u_4^3 + 18\varepsilon^4u_1u_4^4,\end{aligned}$$

$$\begin{aligned}\dot{u}_2 = &-\frac{1}{2}\varepsilon r_2^2 - \frac{411}{160}\varepsilon^3r_2^4 - 2\varepsilon^2r_2^2u_4 - \frac{503}{800}\varepsilon^4r_2^4u_4 - \varepsilon u_4^2 + \frac{33}{8}\varepsilon^3r_2^2u_4^2 + 2\varepsilon^2u_4^3\\ &+\frac{1013}{40}\varepsilon^4r_2^2u_4^3 + 5\varepsilon^3u_4^4 - \frac{11}{2}u_4^5,\end{aligned}$$

$$\begin{aligned}\dot{r}_1 = &-r_1 - \frac{1}{5}\varepsilon^2r_1r_2^2 - \frac{1}{20}\varepsilon^2\cos 2(\theta_1-\theta_2)r_1r_2^2 + \frac{67}{61200}\varepsilon^4r_1r_2^4\\ &-\frac{1563767}{244800}\varepsilon^4\cos 2(\theta_1-\theta_2)r_1r_2^4 - \frac{9}{40}\varepsilon^2\sin 2(\theta_1-\theta_2)r_1r_2^2\\ &-\frac{288671}{27200}\varepsilon^4\sin 2(\theta_1-\theta_2)r_1r_2^4 + \frac{137}{120}\varepsilon^3\cos(\theta_1-\theta_2)r_2^3u_1\\ &-\frac{91}{120}\varepsilon^3\sin(\theta_1-\theta_2)r_2^3u_1 - \frac{82}{45}\varepsilon^3r_1r_2^2u_4 - \frac{73}{100}\varepsilon^3\cos 2(\theta_1-\theta_2)r_1r_2^2u_4\\ &-\frac{137}{75}\varepsilon^3\sin 2(\theta_1-\theta_2)r_1r_2^2u_4 + \varepsilon^2\cos(\theta_1-\theta_2)r_2u_1u_4 - \frac{727}{150}\varepsilon^4\cos(\theta_1-\theta_2)r_2^3u_1u_4\\ &+3\varepsilon^2\sin(\theta_1-\theta_2)r_2u_1u_4 + \frac{9469}{360}\varepsilon^4\sin(\theta_1-\theta_2)r_2^3u_1u_4 + \frac{3881}{900}\varepsilon^4r_1r_2^2u_4^2\\ &-\frac{35947}{3000}\varepsilon^4\cos 2(\theta_1-\theta_2)r_1r_2^2u_4^2 - \frac{16211}{12000}\sin 2(\theta_1-\theta_2)r_1r_2^2u_4^2\\ &-\frac{69}{10}\varepsilon^3\cos(\theta_1-\theta_2)r_2u_1u_4^2 + \frac{97}{10}\varepsilon^3\sin(\theta_1-\theta_2)r_2u_1u_4^2 - \frac{3}{5}\varepsilon^3r_1u_4^3\\ &-\frac{773}{20}\varepsilon^4\cos(\theta_1-\theta_2)r_2u_1u_4^3 - \frac{933}{20}\varepsilon^4\sin(\theta_1-\theta_2)r_2u_1u_4^3 + \frac{519}{50}\varepsilon^4r_1u_4^4,\end{aligned}$$

$$\begin{aligned}\dot{\theta}_1 = &1 + \frac{49}{60}\varepsilon^2r_2^2 - \frac{9}{40}\varepsilon^2\cos 2(\theta_1-\theta_2)r_2^2 + \frac{567587}{108800}\varepsilon^4r_2^4 - \frac{288671}{272000}\varepsilon^4\cos 2(\theta_1-\theta_2)r_2^4\\ &+\frac{1}{20}\varepsilon^2r_2^2\sin 2(\theta_1-\theta_2) + \frac{1563767}{2448000}\varepsilon^4\sin 2(\theta_1-\theta_2)r_2^4 - \frac{91}{120r_1}\varepsilon^3\cos(\theta_1-\theta_2)r_2^3u_1\\ &-\frac{137}{120r_1}\varepsilon^3\sin(\theta_1-\theta_2)r_2^3u_1 + \frac{1}{10}\varepsilon^3r_2^2u_4 - \frac{137}{75}\varepsilon^3\cos 2(\theta_1-\theta_2)r_2^2u_4\end{aligned}$$

$$
\begin{aligned}
&+\frac{73}{100}\varepsilon^3 \sin 2(\theta_1-\theta_2)r_2^2u_4+\frac{3}{r_1}\varepsilon^2\cos(\theta_1-\theta_2)r_2u_1u_4+\frac{9469}{360r_1}\varepsilon^4\cos(\theta_1-\theta_2)r_2^3u_1u_4\\
&-\frac{1}{r_1}\varepsilon^2\sin(\theta_1-\theta_2)r_2u_1u_4+\frac{727}{r_1 150}\varepsilon^4\sin(\theta_1-\theta_2)r_2^3u_1u_4-\frac{5}{2}\varepsilon^2u_4^2\\
&-\frac{50503}{3600}\varepsilon^4r_2^2u_4^2-\frac{16211}{12000}\varepsilon^4\cos 2(\theta_1-\theta_2)r_2^2u_4^2+\frac{35947}{3000}\varepsilon^4\sin 2(\theta_1-\theta_2)r_2^2u_4^2\\
&+\frac{97}{10r_1}\varepsilon^3\cos(\theta_1-\theta_2)r_2u_1u_4^2+\frac{69}{10r_1}\varepsilon^3\sin(\theta_1-\theta_2)r_2u_1u_4^2-\frac{11}{5}\varepsilon^3u_4^3\\
&-\frac{933}{20r_1}\varepsilon^4\cos(\theta_1-\theta_2)r_2u_1u_4^3+\frac{773}{20r_1}\varepsilon^4\sin(\theta_1-\theta_2)r_2u_1u_4^3+\frac{1341}{100}\varepsilon^4u_4^4,
\end{aligned}
$$

$$
\begin{aligned}
\dot r_2 = &-\frac{1}{40}\varepsilon^2r_2^3-\frac{75597}{136000}\varepsilon^4r_2^5+\varepsilon r_2u_4+\frac{83}{50}\varepsilon^3r_2^3u_4+\frac{1}{2}\varepsilon^2r_2u_4^2+\frac{7423}{1000}\varepsilon^4r_2^3u_4^2\\
&-\frac{53}{20}\varepsilon^3r_2u_4^3+\frac{759}{100}\varepsilon^4r_2u_4^4,
\end{aligned}
$$

$$
\begin{aligned}
\dot\theta_2 = 1&-\frac{67}{40}\varepsilon^2r_2^2-\frac{9022679}{816000}\varepsilon^4r_2^4+\frac{7}{100}\varepsilon^3r_2^2u_4+\frac{3}{2}\varepsilon^2u_4^2+\frac{192391}{6000}\varepsilon^4r_2^2u_4^2\\
&+\frac{33}{10}\varepsilon^3u_4^3-\frac{219}{50}\varepsilon^4u_4^4.
\end{aligned}
$$

5.3. Example 3

Calculate the fifth order normal form of the following equation:

$$
\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\\ \dot x_5\\ \dot x_6\\ \dot x_7\end{bmatrix}
=
\begin{bmatrix}
0&1&0&0&0&0&0\\
-1&0&0&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&-1&0&0&0&0\\
0&0&0&0&-1&0&0\\
0&0&0&0&0&-1&1\\
0&0&0&0&0&-1&-1
\end{bmatrix}
\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{bmatrix}
+\varepsilon
\begin{bmatrix}
\varepsilon(x_1^3-2x_1x_3^2-x_1^2x_5+5x_2x_6^2)\\
\varepsilon(x_3^3-2x_2^2x_4+x_1x_5x_6)\\
\varepsilon(x_1^2x_3+3x_1^2x_4)\\
\varepsilon(4x_3^3-x_3^2x_4)\\
(x_1-x_5)^2\\
(x_1-x_4)^2\\
(x_2-x_6)^2
\end{bmatrix}.
$$

The Jacobian matrix evaluated at the origin has the eigenvalues $\pm i$, $\pm i$, -1 and $-1\pm i$. The normal form has a 4-dimensional central manifold. Two of the frequencies are equal; $\omega_1=\omega_2=1$. This is called internal 1:1 main resonance.

The input file is:

```
n=7;
norder=5;
f[1]=x[2]+epsilon^2 (x[1]^3-2 x[1] x[3]^2-x[1]^2 x[5]+5 x[2] x[6]^2);
f[2]=-x[1]+epsilon^2 (x[3]^3-2 x[2]^2 x[4]+x[1] x[5] x[6]);
```

```
f[3]=x[4]+epsilon^2 (x[1]^2 x[3]+^3 x[1]^2 x[4]);
f[4]=-x[3]+epsilon^2 (4 x[3]^3-x[3]^2 x[4]);
f[5]=-x[5]+epsilon (x[1]-x[5])^2;
f[6]=-x[6]+x[7]+epsilon (x[1]-x[4])^2;
f[7]=-x[6]-x[7]+epsilon (x[2]-x[6])^2;
```

The normal form in polar co-ordinates is:

$$\begin{aligned}\dot{u}_1 = & -u_1 + \frac{3}{8}\varepsilon^4\cos(\theta_2-\theta_3)r_2^3r_3u_1 + \varepsilon^2 r_3^2u_1 + \frac{1}{20}\varepsilon^4 r_2^2r_3^2u_1 - \frac{11}{200}\varepsilon^4 r_2^2r_3^2u_1\\ & + \frac{1}{50}\varepsilon^4\cos(\theta_2-\theta_3)r_2r_3^3u_1 + \frac{213}{200}\varepsilon^4 r_3^4u_1 - \frac{6}{25}\varepsilon^4\sin 2(\theta_2-\theta_3)r_2^2r_3^2u_1\\ & + \frac{23}{50}\varepsilon^4\sin(\theta_2-\theta_3)r_2r_3^3u_1,\end{aligned}$$

$$\begin{aligned}\dot{r}_1 = & -r_1 + \left(\frac{1}{10}\varepsilon^2 r_1r_2^2 - \frac{129}{1600}\varepsilon^4 r_1r_2^4 - \frac{32501}{122400}\varepsilon^4 r_1r_2^2r_3^2\right)\cos 2(\theta_1-\theta_2)\\ & + \left(-\frac{3}{20}\varepsilon^2 r_1r_2r_3 - \frac{493}{800}\varepsilon^4 r_1r_2^3r_3 - \frac{88079}{163200}\varepsilon^4 r_1r_2r_3^3\right)\cos(2\theta_1-\theta_2-\theta_3)\\ & + \left(\frac{3}{4}\varepsilon^4 r_1r_2^3r_3 + \frac{1829}{1200}\varepsilon^4 r_1r_2r_3^3\right)\cos(\theta_2-\theta_3) - \frac{89}{1600}\varepsilon^4 r_1r_2^3r_3\cos(2\theta_1-3\theta_2+\theta_3)\\ & + \left(-\frac{1}{20}\varepsilon^2 r_1r_3^2 + \frac{1127}{3200}\varepsilon^4 r_1r_2^2r_3^2 + \frac{3181}{12240}\varepsilon^4 r_1r_3^4\right)\cos 2(\theta_1-\theta_3) + \frac{53}{72}\varepsilon^4 r_1r_2^2r_3^2\\ & - \frac{73}{144}\varepsilon^4 r_1r_2^2r_3^2\cos 2(\theta_2-\theta_3) + \frac{9}{320}\varepsilon^4 r_1r_2r_3^3\cos(2\theta_1+\theta_2-3\theta_3)\\ & + \frac{401}{1200}\varepsilon^4 r_1r_3^4 + \left(\frac{3}{40}\varepsilon^2 r_1r_2^2 + \frac{161}{800}\varepsilon^4 r_1r_2^4 - \frac{1361}{10200}\varepsilon^4 r_1r_2^2r_3^2\right)\sin 2(\theta_1-\theta_2)\\ & + \frac{849}{3200}\varepsilon^4 r_1r_2r_3^3\sin(2\theta_1+\theta_2-3\theta_3) + \left(\frac{2}{5}\varepsilon^2 r_1r_3^2 - \frac{2601}{3200}\varepsilon^4 r_1r_2^2r_3^2\right.\\ & \left. - \frac{19081}{244800}\varepsilon^4 r_1r_3^4\right)\sin 2(\theta_1-\theta_3) + \frac{13}{16}\varepsilon^4 r_1r_2^2r_3^2\sin 2(\theta_2-\theta_3)\\ & + \left(\frac{1}{5}\varepsilon^2 r_1r_2r_3 - \frac{11}{50}\varepsilon^4 r_1r_2^3r_3 - \frac{289639}{489600}\varepsilon^4 r_1r_2r_3^3\right)\sin(2\theta_1-\theta_2-\theta_3)\\ & + \left(\frac{7}{24}\varepsilon^4 r_1r_2^3r_3 + \frac{287}{200}\varepsilon^4 r_1r_2r_3^3\right)\sin(\theta_2-\theta_3) - \frac{21}{3200}\varepsilon^4 r_1r_2^3r_3\sin(2\theta_1-3\theta_2+\theta_3)\\ & - \frac{1}{10}\varepsilon^3 r_2r_3^2u_1\cos(\theta_1+\theta_2-2\theta_3) + \frac{3}{5}\varepsilon^3 r_3^3u_1\cos(\theta_1-\theta_3)\\ & - \frac{1}{4}\varepsilon^3 r_2r_3^2u_1\sin(\theta_1-\theta_2) + \frac{3}{40}\varepsilon^3 r_2r_3^2u_1\sin(\theta_1+\theta_2-2\theta_3) + \frac{1}{20}\varepsilon^3 r_3^3u_1\sin(\theta_1-\theta_3).\end{aligned}$$

$$\dot{\theta}_1 = 1 - \frac{1}{4}\varepsilon^2 r_2^2 + \left(\frac{3}{40}\varepsilon^2 r_2^2 + \frac{161}{800}\varepsilon^4 r_2^4 - \frac{1361}{10200}\varepsilon^4 r_2^2 r_3^2\right)\cos 2(\theta_1 - \theta_2)$$
$$+ \frac{75}{256}\varepsilon^4 r_2^4 + \left(\frac{1}{5}\varepsilon^2 r_2 r_3 - \frac{11}{50}\varepsilon^4 r_2^3 r_3\right)\cos(2\theta_1 - \theta_2 - \theta_3)$$
$$+ \left(\frac{71}{96}\varepsilon^4 r_2^3 r_3 - \frac{181}{2400}\varepsilon^4 r_2 r_3^3\right)\cos(\theta_2 - \theta_3) - \frac{21}{3200}\varepsilon^4 r_2^3 r_3 \cos(2\theta_1 - 3\theta_2 + \theta_3) +$$
$$- \frac{5}{6}\varepsilon^2 r_3^2 + \left(\frac{2}{5}\varepsilon^2 r_3^2 - \frac{2601}{3200}\varepsilon^4 r_2^2 r_3^2 - \frac{19081}{43200}\varepsilon^4 r_3^4\right)\cos 2(\theta_1 - \theta_3) - \frac{1261}{2880}\varepsilon^4 r_2^2 r_3^2$$
$$- \frac{59}{480}\varepsilon^4 r_2^2 r_3^2 \cos 2(\theta_2 - \theta_3) + \frac{849}{3200}\varepsilon^4 r_2 r_3^3 \cos(2\theta_1 + \theta_2 - 3\theta_3)$$
$$- \frac{289639}{489600}\varepsilon^4 r_2 r_3^3 \cos(2\theta_1 - \theta_2 - \theta_3) - \frac{2287}{43200}\varepsilon^4 r_3^4$$
$$+ \left(-\frac{1}{10}\varepsilon^2 r_2^2 + \frac{129}{1600}\varepsilon^4 r_2^4 + \frac{32501}{122400}\varepsilon^4 r_2^2 r_3^2\right)\sin 2(\theta_1 - \theta_2)$$
$$- \frac{9}{320}\varepsilon^4 r_2 r_3^3 \sin(2\theta_1 + \theta_2 - 3\theta_3) + \frac{823}{2880}\varepsilon^4 r_2^2 r_3^2 \sin 2(\theta_2 - \theta_3)$$
$$+ \left(\frac{1}{20}\varepsilon^2 r_3^2 - \frac{1127}{3200}\varepsilon^4 r_2^2 r_3^2 - \frac{3181}{122400}\varepsilon^4 r_3^4\right)\sin 2(\theta_1 - \theta_3)$$
$$+ \left(\frac{3}{20}\varepsilon^2 r_2 r_3 + \frac{493}{800}\varepsilon^4 r_2^3 r_3 + \frac{88079}{163200}\varepsilon^4 r_2 r_3^3\right)\sin(2\theta_1 - \theta_2 - \theta_3)$$
$$+ \left(-\frac{1}{2}\varepsilon^2 r_2 r_3 + \frac{27}{64}\varepsilon^4 r_2^3 r_3 - \frac{1429}{7200}\varepsilon^4 r_2 r_3^3\right)\sin(\theta_2 - \theta_3)$$
$$+ \frac{89}{1600}\varepsilon^4 r_2^3 r_3 \sin(2\theta_1 - 3\theta_2 + \theta_3) - \frac{1}{4r_1}\varepsilon^3 r_2 r_3^2 u_1 \cos(\theta_1 - \theta_2)$$
$$+ \frac{3}{40r_1}\varepsilon^3 r_2 r_3^2 u_1 \cos(\theta_1 + \theta_2 - 2\theta_3) + \frac{1}{20r_1}\varepsilon^3 r_3^3 u_1 \cos(\theta_1 - \theta_3)$$
$$+ \frac{1}{10r_1}\varepsilon^3 r_2 r_3^2 u_1 \sin(\theta_1 + \theta_2 - 2\theta_3) - \frac{3}{5r_1}\varepsilon^3 r_3^3 u_1 \sin(\theta_1 - \theta_3),$$

$$\dot{r}_2 = -\frac{1}{8}\varepsilon^2 r_2^3 - \frac{1}{8}\varepsilon^4 r_2^5 + \left(\frac{17}{128}\varepsilon^4 r_2^4 r_3 - \frac{15}{64}\varepsilon^4 r_2^2 r_3^3\right)\cos(\theta_2 - \theta_3) + \frac{1}{4}\varepsilon^2 r_2 r_3^2$$
$$+ \left(\frac{1}{8}\varepsilon^2 r_2 r_3^2 + \frac{7}{128}\varepsilon^4 r_2^3 r_3^2 - \frac{39}{128}\varepsilon^4 r_2 r_3^4\right)\cos 2(\theta_2 - \theta_3) + \frac{9}{64}\varepsilon^4 r_2^3 r_3^2$$
$$- \frac{3}{32}\varepsilon^4 r_2^2 r_3^3 \cos 3(\theta_2 - \theta_3) - \frac{1}{32}\varepsilon^4 r_2^2 r_3^3 \sin 3(\theta_2 - \theta_3)$$

$$+\left(-\frac{3}{8}\varepsilon^2 r_2 r_3^2+\frac{35}{128}\varepsilon^4 r_2^3 r_3^2+\frac{27}{128}\varepsilon^4 r_2 r_3^4\right)\sin 2(\theta_2-\theta_3)$$
$$-\left(\frac{21}{128}\varepsilon^4 r_2^4 r_3+\frac{1}{64}\varepsilon^4 r_2^2 r_3^3\right)\sin(\theta_2-\theta_3),$$

$$\dot\theta_2=1-\frac{3}{2}\varepsilon^2 r_2^2-\frac{827}{256}\varepsilon^4 r_2^4+\left(\frac{21}{128}\varepsilon^4 r_2^3 r_3-\frac{5}{64}\varepsilon^4 r_2 r_3^3\right)\cos(\theta_2-\theta_3)$$
$$+\frac{3}{4}\varepsilon^2 r_3^2+\left(-\frac{3}{8}\varepsilon^2 r_3^2+\frac{37}{64}\varepsilon^4 r_2^2 r_3^2+\frac{27}{128}\varepsilon^4 r_3^4\right)\cos 2(\theta_2-\theta_3)-\frac{145}{64}\varepsilon^4 r_2^2 r_3^2$$
$$-\frac{1}{32}\varepsilon^4 r_2 r_3^3\cos 3(\theta_2-\theta_3)-\frac{45}{128}\varepsilon^4 r_3^4+\frac{3}{32}\varepsilon^4 r_2 r_3^3\sin 3(\theta_2-\theta_3)$$
$$+\left(-\frac{1}{8}\varepsilon^2 r_3^2-\frac{1}{64}\varepsilon^4 r_2^2 r_3^2+\frac{39}{128}\varepsilon^4 r_3^4\right)\sin 2(\theta_2-\theta_3)$$
$$+\left(-\frac{7}{128}\varepsilon^4 r_2^3 r_3+\frac{27}{64}\varepsilon^4 r_2 r_3^3\right)\sin(\theta_2-\theta_3),$$

$$\dot r_3=\left(-\frac{37}{256}\varepsilon^4 r_2^5-\frac{3}{4}\varepsilon^2 r_2 r_3^2+\frac{93}{256}\varepsilon^4 r_2^3 r_3^2-\frac{159}{640}\varepsilon^4 r_2 r_3^4\right)\cos(\theta_2-\theta_3)-\frac{1}{2}\varepsilon^2 r_2^2 r_3$$
$$+\left(-\frac{1}{4}\varepsilon^2 r_2^2 r_3-\frac{9}{64}\varepsilon^4 r_2^4 r_3+\frac{21}{40}\varepsilon^4 r_2^2 r_3^3\right)\cos 2(\theta_2-\theta_3)-\frac{9}{16}\varepsilon^4 r_2^4 r_3$$
$$+\frac{3}{256}\varepsilon^4 r_2^3 r_3^2\cos 3(\theta_2-\theta_3)+\frac{3}{8}\varepsilon^2 r_3^3-\frac{239}{320}\varepsilon^4 r_2^2 r_3^3$$
$$+\left(-\frac{7}{64}\varepsilon^4 r_2^4 r_3-\frac{171}{320}\varepsilon^4 r_2^2 r_3^3\right)\sin 2(\theta_2-\theta_3)$$
$$+\left(\frac{3}{8}\varepsilon^2 r_2^3+\frac{51}{64}\varepsilon^4 r_2^5+\frac{29}{256}\varepsilon^4 r_2^3 r_3^2-\frac{189}{320}\varepsilon^4 r_2 r_3^4\right)\sin(\theta_2-\theta_3),$$

$$\dot\theta_3=1+\frac{3}{32}\varepsilon^4 r_2^4+\left(\frac{7}{64}\varepsilon^4 r_2^4+\frac{1}{800}\varepsilon^4 r_2^2 r_3^2\right)\cos 2(\theta_2-\theta_3)$$
$$+\left(-\frac{3}{8r_3}\varepsilon^2 r_2^3-\frac{51}{64r_3}\varepsilon^4 r_2^5+\frac{69}{256}\varepsilon^4 r_2^3 r_3+\frac{147}{1600}\varepsilon^4 r_2 r_3^3\right)\cos(\theta_2-\theta_3)$$
$$+\frac{313}{320}\varepsilon^4 r_2^2 r_3^2+\frac{461}{6400}\varepsilon^4 r_3^4+\frac{3}{256}\varepsilon^4 r_2^3 r_3\sin 3(\theta_2-\theta_3)$$
$$+\left(-\frac{1}{4}\varepsilon^2 r_2^2-\frac{9}{64}\varepsilon^4 r_2^4+\frac{109}{400}\varepsilon^4 r_2^2 r_3^2\right)\sin 2(\theta_2-\theta_3)$$
$$+\left(-\frac{37}{256r_3}\varepsilon^4 r_2^5-\frac{1}{4}\varepsilon^2 r_2 r_3+\frac{243}{256}\varepsilon^4 r_2^3 r_3+\frac{2677}{3200}\varepsilon^4 r_2 r_3^3\right)\sin(\theta_2-\theta_3).$$

6. Conclusion

A simple and efficient symbolic computer program using MATHEMATICA language has been developed for computing the high-dimension and high order normal forms without central manifold reduction for a system in semi-simple case. Explicit formulas for the normal form and the associated non-linear transformation are given in terms of the coefficients of the original differential equation. The program can be easily extended to calculate high-dimensional normal forms.

Acknowledgements

The research is supported by a Royal Society K.C. Wong Fellowship to Qichang Zhang to visit Manchester to complete this work.

Appendix A

```
Print["****************************************************"];
Print["           Read the input functions"                ];
Print["****************************************************"];
SetDirectory["path"]
ReadList["file"];
Print["****************************************************"];
Print["Transform the real functions form to complex functions form"];
Print["****************************************************"];
epsilon=\[Epsilon];
FA=Table[0,{i,n}];
F=Table[f[i],{i,1,n}];
X=Table[x[i],{i,1,n}];
U=Table[u[i],{i,1,n}];
Jacobi[funs_List,vars_List]:=Outer[D,funs,vars]
A=Jacobi[F,X];
Do[A=A/.{x[k]->0};,{k,1,n}]
B=Eigenvalues[A]
Table[bb[k]=0,{k,1,n}];
Do[
   If[B[[k]]!=0,
      bb[k]=1/2;,
      bb[k]=1;];
,{k,1,n}];
```

```
BC=DiagonalMatrix[Table[bb[k],{k,1,n}]];
EV=Eigenvectors[A]
EV=Transpose[EV].BC;
IEV=Inverse[EV];
X=EV.U;
Do[x[i]=X[[i]],{i,n}]
F=IEV.F;
J0=DiagonalMatrix[B];
F=F-J0.U;
Do[
  FFI[i]=CoefficientList[F[[i]],\[Epsilon]];
  le[i]=Length[FFI[i]];
  Do[
    fci[i,k1]=Part[FFI[i],k1];
    Do[
      fci[i,k1]=fci[i,k1]/.{u[k]->v[k]};
      ,{k,1,n}];
    ,{k1,1,le[i]}];
  ,{i,1,n}];
Print["****************************************************"];
Print["Compute the normal form and nonlinear transformation"];
Print["****************************************************"];
Co3[i_]:=D[b[k,nn],{u[i],nk[i]}]/nk[i]!
ne[k_]:=j-Sum[nk[i],{i,1,k-1}];
Table[ne[i]=0,{i,n,8}];
Table[nk[i]=0,{i,1,n}];
Do[TP[j]=Table[0,{i,1,n}],{j,0,norder-1}];
Do[TNF[j]=Table[0,{i,1,n}],{j,0,norder-1}];
Do[
 If[j>2,
  Do[
    Do[
      fc[k,k1]=fci[k,k1];
      If[k1<j,
        Do[vv[k]=u[k]+Sum[\[Epsilon]^j1* TP[j1][[k]],{j1,1,j-k1}],{k,1,n}];
        Do[fc[k,k1]=fc[k,k1]/.{v[i]->vv[i]};,{i,1,n}];,
        Do[fc[k,k1]=fc[k,k1]/.{v[i]->u[i]};,{i,1,n}];
        ];
    ,{k1,2,le[k]}];
    FF[k]=Table[fc[k,k1],{k1,2,le[k]}];
    EE=Table[If[i<j,\[Epsilon]^i,0],{i,1,le[k]-1}];
    f[k]=FF[k].EE;
  ,{k,1,n}];
```

```
    F=Table[f[i],{i,1,n}];
    Do[FF[i]=CoefficientList[F[[i]],\[Epsilon]],{i,1,n}];,
    Do[FF[i]=FFI[i],{i,1,n}]
  ];
  Do[le1[k]=Length[FF[k]],{k,1,n}];
  Do[If[j>le1[k],g[k]=0;,g[k]=Part[FF[k],j]];,{k,1,n}];
  FA=Sum[Jacobi[TP[j-1-j1],U].TNF[j1],{j1,1,j-2}];
  If[FA==0,FA=Table[0,{i,n}];];
  Table[p[k]=0,{k,n}];
  Table[nf[k]=0,{k,n}];
  Do[
        Do[
                    Do[
                    Do[
                    Do[
                            Do[
                                    Do[
                                            Do[
                                                    nk[n]=j-Sum[nk[i],{i,1,n-1}];
                                                    Do[
                                       nn=Sum[nk[i]*10^(n-i),{i,1,n}]+10^n;
                                       b[k,nn]=g[k]-FA[[k]];
                                       Do[b[k,nn]=Co3[i],{i,1,n}];
                                       \[Lambda]=Sum[B[[i]]*nk[i],{i,1,n}]-B[[k]];
                                       If[
                             \[Lambda]==0,
                             un=b[k,nn];
                             hp=0;,
                             hp=b[k,nn]/\[Lambda];
                             un=0
                                            ];
                                               xs=Product[u[i]^nk[i],{i,1,n}];
                                               p[k]=p[k]+hp*xs;
                                               nf[k]=nf[k]+un*xs;
                                                    ,{k,1,n}];
                                            ,{nk[8],0,ne[8]}];
                             ,{nk[7],0,ne[7]}];
                    ,{nk[6],0,ne[6]}];
             ,{nk[5],0,ne[5]}];
          ,{nk[4],0,ne[4]}];
       ,{nk[3],0,ne[3]}];
   ,{nk[2],0,ne[2]}];
,{nk[1],0,j}];
TP[j-1]=Table[p[k],{k,1,n}];
```

```
TNF[j-1]=Table[nf[k],{k,1,n}];
,{j,2,norder}];
"**************************************************">>d.txt;
"      The normal form in complex form              ">>>d.txt;
"**************************************************">>>d.txt;
Do[tf[k]=Sum[\[Epsilon]^j TNF[j],{j,1,norder-1}][[k]],{k,1,n}];
Do[
  "D[u["<>ToString[i]<>"],t]=">>>d.txt;
  tf[i]=B[[i]]*u[i]+tf[i]>>>d.txt;
,{i,1,n}];
"**************************************************">>>d.txt;
"    Transform back to system in real form                    ">>>d.txt;
"**************************************************">>>d.txt;
Clear[F,f,X,x,Y,y,v];
X=Table[x[i],{i,1,n}];
F=Simplify[Table[tf[i],{i,1,n}]]
Do[F=F/.{u[k]->y[k]};,{k,1,n}];
Y=IEV.X;
Do[y[k]=Y[[k]],{k,1,n}];
"D[Y,t]=">>>d.txt
F1=Simplify[EV.F]>>>d.txt
"*****************************************************">>>d.txt;
"    Transform back to system in Polar form        ">>>d.txt;
"*****************************************************">>>d.txt;
conjug=Complex[a_,b_]:->Complex[a,-b];
nz=0;ni=0;cnz=0;cni=0;cm=0;
Do[
   If[Re[B[[k]]] ≠ 0,
      If[Im[B[[k]]]==0,cnz=cnz+1;];,
      cm=cm+1;
      If[Im[B[[k]]]==0,nz=nz+1;];];
   ,{k,n}];
sm=n-cm;
ni=(cm-nz)/2;
cni=(cm-cnz)/2;
k2=0;k1=0;
Do[
   If[Im[B[[k]]]>0,
      k2=k2+1;
      v[k]=r[k2] Exp[I θ[k2]];,
      If[Im[B[[k]]]==0,
         v[k]=u[k];,
         k1=k1+1;
         v[k]=r[k1] Exp[-I θ[k1]];];];
   ,{k,n}];
k1=0;k2=0;nzn=nz+cnz;
```

```
Do[
 If[Im[B[[k]]]==0,
    ga=tf[k];
    Do[ga=ga/.{u[i]->v[i]};,{i,1,n}];
    k2=k2+1;
    "D[u["<>ToString[k2]<>"],t]=">>>d.txt;
    g[k2]=ga>>>d.txt;,
    If[Im[B[[k]]]<0,
      ga=tf[k];
      Do[ga=ga/.{u[i]->v[i]};,{i,1,n}];
      k1=k1+1;
      gc=ga Exp[I θ[k1]];
      gd=gc/.conjug;
      "D[r["<>ToString[k1]<>"],t]=">>>d.txt;
      g[nzn+2 k1-1]= Expand[TrigReduce[ComplexExpand[Simplify
          [(gc+gd)/2]]]]>>>d.txt;
      "D[q["<>ToString[k1]<>"],t]=">>>d.txt;
          g[nzn+2 k1]= Expand[TrigReduce[ComplexExpand[Simplify
        [-(gc-gd)/(2*I* r[k1])]]]]>>>d.txt;
    ];
  ];
,{k,n}]
```

References

[1] G. Kern, A. Maitz, Normal form transformation and an application to a flutter-type of vibration, International Journal of Nonlinear Mechanics 33 (5) (1998) 741–751.

[2] A. Khajepour, M.F. Golnaraghi, K.A. Morris, Modal coupling controller design using a normal form method 0.1. Dynamics, Journal of Sound and Vibration 205 (5) (1997) 657–670.

[3] M.H. Fredriksson, A.B. Nordmark, On normal form calculations in impact oscillators, Proceedings of the Royal Society London A 456 (2000) 315–329.

[4] P. Yu, Q.S. Bi, Analysis of non-linear dynamics and bifurcations of a double pendulum, Journal of Sound and Vibration 217 (4) (1998) 691–736.

[5] P. Yu, W. Zhang, Q.S. Bi, Vibration analysis on a thin plate with the aid of computation of normal forms, International Journal of Nonlinear Mechanics 36 (4) (2001) 597–627.

[6] D. Wang, An introduction to the normal form theory of ordinary differential equation, Advances in Mathematics 19 (1989) 38–71.

[7] H. Nayfeh, Method of Normal Form, Wiley, New York, 1993.

[8] P. Yu, Computation of normal forms via a perturbation technique, Journal of Sound and Vibration 211 (1) (1998) 19–38.

[9] A.Y.T. Leung, Q.C. Zhang, Complex normal form for strongly nonlinear vibration systems exemplified by Dufing–Van der Pol equation, Journal of Sound and Vibration 213 (5) (1998) 907–914.

[10] A.Y.T. Leung, Q.C. Zhang, High order normal form and period averaging, Journal of Sound and Vibration 217 (5) (1998) 795–806.

[11] W.Y. Zhang, K. Huseyin, Y.S. Chen, A new approach for obtaining normal forms of non-linear systems, Journal of Sound and Vibration 210 (5) (1998) 609–625.

[12] W.Y. Zhang, K. Huseyin, M. Ye, On the computation of the coefficients associated with high order normal forms, Journal of Sound and Vibration 232 (3) (2000) 525–540.

[13] Y.S. Chen, Q.C. Zhang, A new method of calculating the asymptotic solution of nonlinear vibration system, Acta Mechanica Sinica 22 (1990) 413–419 (in Chinese).
[14] Q.S. Bi, P. Yu, Double Hopf bifurcations and chaos of a nonlinear vibration system, Nonlinear Dynamics 19 (4) (1999) 313–332.
[15] B.D. Hassard, N.D. Karazinoff, Y.-H. Wan, Theory and Applications of Hopf Bifurcation, London Mathematical Society Lecture Note Series 41, London, 1981.
[16] B.D. Hassard, Y.H. Wan, Bifurcation formulae derived from center manifold theory, Journal of Mathematical Analysis and Application 63 (1978) 297–312.
[17] J.M. Guckenheimer, P. Holmes, Nonlinear Oscillation, Dynamical Systems and Bifurcation of Vector Fields, Springer, New York, 1983.
[18] Y.S. Chen, A.Y.T. Leung, Bifurcation and Chaos in Engineering,, Springer, London, 1998.
[19] Yu.N. Bibikov, in: Local Theory of Nonlinear Analytic Ordinary Differential Equations, Lecture Notes in Mathematics, Vol. 702, Springer, New York, 1979.
[20] P. Coullet, E. Spiesel, Amplitude equations for systems with competing instabilities, SIAM Journal of Applied Mathematics 43 (1983) 776–821.
[21] K. Huseyin, W. Zhang, On the normal forms associate with high dimensional systems, Journal of Vibration and Acoustics 123, 157–169.
[22] Z.Q. Wu, The Direct Method for Calculating Normal Form and Nonlinear Model, Thesis, Tianjin University, China, 1993 (in Chinese).